The Picard stack and Div^1

November 11, 2024

1. Vector bundles on analytic stacks

The goal of today's talk is to introduce and study the Picard stack Pic classifying line bundles on the twistor $\mathbb{P}^1$ as introduced last time, as well as the related stack Div^1 (which will be especially important as we continue). This is the first example of the stack Bun_G parametrizing G-bundles on the curve, for the case $G = \mathbb{G}_\mathrm{m}$. First, though, we need to say what this means: what is a vector bundle on an analytic stack?

The most natural approach, parallel to how we define analytic stacks themselves, is to work locally: on an affine analytic stack $\mathrm{AnSpec}\, A$, a vector bundle should be a finite projective A-module, i.e. a perfect complex concentrated in degree 0. We then hope that the functor associating to A the anima of finite projective A-modules descends along !-covers.

Unfortunately, this fails to be true in general. It is true however if we restrict to certain special analytic rings, called Fredholm rings; in particular we will see that totally disconnected $\mathbb{C}$-algebras are Fredholm, and so we get a good notion of vector bundles on analytic stacks which are covered by totally disconnected algebras, and in particular for those such as the relative twistor $\mathbb{P}^1$ which are obtained as realizations of totally disconnected stacks.

Proposition 1. *Let $A \to B$ be a map of analytic rings such that $- \otimes^{\mathbb{L}}_A B : D(A) \to D(B)$ is conservative (e.g. if the map is !-descendable). If M is a static finitely generated $A^{\triangleright}(*)$-module and the (non-derived) tensor product $M \otimes_A B$ vanishes, then $M = 0$.*

Proof. If M is nonzero, we can find a nonzero quotient generated by a single element with the same properties, which is then of the form A/I for some ideal $I \subset A^{\triangleright}(*)$ and so carries an algebra structure; thus if we can prove $A/I = 0$ then we must have $M = 0$. By assumption $A/I \otimes_A B = 0$ and hence (since A/I is static) $A/I \otimes^{\mathbb{L}}_A B = 0$, and so the claim follows since we assume $- \otimes^{\mathbb{L}}_A B$ is conservative. □

Corollary 2. *Let $A \to B$ be a map of analytic rings such that $- \otimes^{\mathbb{L}}_A B$ is conservative and let M be a perfect complex of A-modules. Then the amplitude of M is the same as the amplitude of $M \otimes^{\mathbb{L}}_A B$; in particular if M is a finite projective A-module, then $M \otimes^{\mathbb{L}}_A B$ is a finite projective B-module.*

Proof. It suffices to show that the amplitudes agree on the right. In each degree i, let $M^{(i)} = (\cdots \to M_{i-1} \to M_i) \to M$ be the subcomplex given by truncation; taking $H^0(M/M^{(i)})$ then gives a static finitely generated $A^{\triangleright}(*)$-module, which vanishes for i above the amplitude of M and whose base change to B vanishes above the amplitude of $M \otimes^{\mathbb{L}}_A B$. By the previous proposition, the base change vanishes if and only if the original module does, and so these conditions are equivalent. □

Corollary 3. *Let A be an analytic ring such that any dualizable object of $D(A)$ is a perfect complex. Then any object of $D(A)$ which is !-locally a perfect complex is in fact a perfect complex, and its amplitude can be determined !-locally. In particular any object of $D(A)$ which is !-locally a finite projective module is in fact a finite projective A-module.*

Proof. An object which is !-locally a perfect complex is dualizable, which by our assumption on A implies that it is in fact a perfect complex. The amplitude statement follows from the previous corollary. □

This condition on A will be useful enough that we give it a name: an analytic ring A is Fredholm if every dualizable object of $D(A)$ is a perfect complex.

Proposition 4. *An analytic ring A is Fredholm if and only if for any compact object $K \in D(A)$ and trace-class endomorphism $f : K \to K$, the cone of $1 - f$ is a perfect complex.*

Proof. If the stated condition holds, then any dualizable object K is compact and the identity is trace-class, so by assumption the cone of $1 - 1 : K \to K$, which is (isomorphic to) just K, is a perfect complex. Conversely if A is Fredholm, K is compact, and $f : K \to K$ is trace-class, then the cone of $1 - f$ is equivalently that of $1 - f$ on $K[1/f]$ and hence compact and nuclear, therefore dualizable, and so since A is Fredholm it must in fact be a perfect complex. □

Proposition 5. *Let $A \to \overline{A}$ be a surjective map of analytic rings such that $\overline{A}$ is Fredholm and for every finitely presented static A-module M, if $M \otimes_A \overline{A} = 0$ then $M = 0$. Then A is Fredholm.*

Proof. Assume K is a dualizable A-module. By shifting, we can assume in nonnegative (homological) degrees, with $H_0(K)$ nonzero. By compactness, $H_0(K)$ is finitely presented, and $K \otimes_A \overline{A}$ is dualizable and hence a perfect complex since $\overline{A}$ is Fredholm. Thus it can be killed by finitely many elements: we can find $x_1, \ldots, x_r \in H_0(K)$ such that $M = H_0(K)/(x_1, \ldots, x_r)$ satisfies $M \otimes_A \overline{A} = 0$, and so by assumption $M = 0$, i.e. $H_0(K)$ is a quotient of a free module A^r generated by the x_i. Taking the cone of $A^r \to H_0(K)$ and repeating the process, we get a presentation of K by a chain of free A-modules; by compactness this is perfect, so every dualizable A-module is a perfect complex, i.e. A is Fredholm. □

In particular, one can check Fredholm-ness after passing along the map $A \to \pi_0 A$, and so can reduce to the static case. Our interest is in the bounded case, where we can say more:

Proposition 6. *Let A be a bounded gaseous $\mathbb{R}$-algebra and M be a finitely presented static A-module. If $M \otimes_A A^{\dagger\text{-red}} = 0$, then $M = 0$. In particular if $A^{\dagger\text{-red}}$ is Fredholm, then so is A.*

Proof sketch. The second part follows from the first by the previous proposition, so it suffices to prove the first part. As above, we can assume that A is static. Since M is finitely presented, we can find a surjection $A[S] \to M$, corresponding to a map $S \to M$. The image of $\mathrm{Nil}^\dagger(A)[S] \to A[S] \to M$ is killed after base change to $A^{\dagger\text{-red}}$; since $M \otimes_A A^{\dagger\text{-red}} = 0$ by assumption, the map from $\mathrm{Nil}^\dagger(A)[S]$ is also surjective. Thus, after passing to some cover $S' \to S$, we can lift $S' \to S \to M$ to a map $g : S' \to \mathrm{Nil}^\dagger(A)[S]$. By carefully iterating a lifting process whose details we omit, we can actually assume $S' = S$, and so we obtain $g : S \to \mathrm{Nil}^\dagger(A)[S]$ lifting $S \to M$, which induces an endomorphism $A[S] \to \mathrm{Nil}^\dagger(A)[S] \to A[S]$ which we also denote by g. Since M is a quotient of the cokernel of $1 - g$, to show it is zero it suffices to show that $1 - g$ is surjective; in fact it is an isomorphism, with inverse $1 + g + g^2 + \cdots$ which one can show exists via a similar iterating process. □

Corollary 7. *Every totally disconnected $\mathbb{C}$-algebra is Fredholm.*

Proof. Working via stalks as we often do for totally disconnected $\mathbb{C}$-algebras, the †-reduction is $\mathbb{C}$ and so it suffices to prove that $\mathbb{C}$ is Fredholm, i.e. all dualizable objects in gaseous $\mathbb{C}$-vector spaces are perfect complexes, which is the same argument as the usual one that dualizable objects in $D(\mathbb{C})$ are perfect complexes. □

In particular we have shown that finite projective modules satisfy good descent properties on Fredholm rings and therefore on totally disconnected $\mathbb{C}$-algebras. We can now define vector bundles as expected: a vector bundle on an analytic stack X is an object $E \in D(X)$ such that there is some !-cover by affine analytic stacks on which E pulls back to finite projective modules. For realizations of totally disconnected stacks, these have the descent properties we expect.

2. The Picard stack

We can now define the Picard stack Pic straightforwardly as the functor sending a totally disconnected $\mathbb{C}$-algebra A to the anima of line bundles on $X_{\mathbb{R},A}$.

Proposition 8. *The Picard stack* Pic *is an object of* TDStack*.*

Proof. One cannot apply naive descent arguments, as $A \mapsto X_{\mathbb{R},A}$ doesn't obviously preserve !-covers and does not commute with finite limits. However, by definition a line bundle on $X_{\mathbb{R},A}$ is equivalently a pair of line bundles on $\operatorname{AnSpec} A$ and $X_{\mathbb{R}} \times_{\operatorname{AnSpec} R} \operatorname{AnSpec} \operatorname{Cont}(S, \mathbb{R})$ together with an isomorphism of their pullbacks to $\operatorname{AnSpec} \operatorname{Cont}(S, \mathbb{R})$. By the results of the previous section, since A is a totally disconnected $\mathbb{C}$-algebra it is Fredholm and so vector bundles on it have good descent properties; for the other parts, it suffices to see that the functor mapping S to the category of vector bundles on $\operatorname{AnSpec} \operatorname{Cont}(S, \mathbb{R})$ is a stack, which is the lemma below.

The careful reader will note that to make sure this plays well with passing to strongly totally disconnected stacks we should verify that the moduli problem commutes with $\aleph_1$-filtered colimits. This has to do with sequential colimits preserving Cauchy completeness, which we leave to reader more careful than the author to verify. □

Lemma 9. *The functor sending a compact Hausdorff set S to the category of vector bundles on* $\operatorname{AnSpec} \operatorname{Cont}(S, \mathbb{R})$ *is a sheaf of categories for the topology where covers are jointly surjective finite families.*

This is a "pure condensed math" result, generalizing Theorem 3.3 of Condensed.pdf, and so we omit the proof. Notably this is stronger than what we needed above, which is just the restriction to light profinite sets.

For each integer n, we have a line bundle $\mathcal{O}_{X_{\mathbb{R}}}(n)$ on the (absolute) twistor $\mathbb{P}^1$, which pulls back to a line bundle $\mathcal{O}_{X_{\mathbb{R},A}}(n)$ on $X_{\mathbb{R},A}$ with endomorphisms the invertible elements in

$$\operatorname{Hom}(\mathcal{O}_{X_{\mathbb{R},A}}(n), \mathcal{O}_{X_{\mathbb{R},A}}(n)) \simeq \operatorname{Hom}(\mathcal{O}_{X_{\mathbb{R},A}}, \mathcal{O}_{X_{\mathbb{R},A}}) = \mathcal{BC}(\mathcal{O}) \simeq \mathbb{R}^{\mathrm{la}}$$

as last time, i.e. $\operatorname{Aut}(\mathcal{O}_{X_{\mathbb{R},A}}(n)) \simeq \mathbb{R}^{\times,\mathrm{la}}$. This gives for each n a map $*/\mathbb{R}^{\times,\mathrm{la}} \to \operatorname{Pic}$, which assemble to a map

$$\bigsqcup_{n \in \mathbb{Z}} */\mathbb{R}^{\times,\mathrm{la}} \to \operatorname{Pic}.$$

Theorem 10. *The map*

$$\coprod_{n\in\mathbb{Z}} */\mathbb{R}^{\times,\mathrm{la}} \to \mathrm{Pic}$$

is an isomorphism.

In particular there is a degree map $\deg : \mathrm{Pic} \to \mathbb{Z}$ given by composing the inverse of the map above with each structure map $*/\mathbb{R}^{\times,\mathrm{la}} \to *$, sending $\mathcal{O}_{X_\mathbb{R}}(n) \mapsto n$.

Proof. The existence of the line bundles $\mathcal{O}_{X_{\mathbb{R},A}}(n)$ and the fact that their automorphism groups are $\mathbb{R}^{\times,\mathrm{la}}$ are straightforward, giving the injectivity of the map; the hard part is the surjectivity, i.e. the statement that any line bundle on $X_{\mathbb{R},A}$ is isomorphic to one of these.

Let $\mathcal{L}$ be a line bundle on $X_{\mathbb{R},A}$, which by definition gives rise to a line bundle $\overline{\mathcal{L}}$ on $X_\mathbb{R} \times_{\mathrm{AnSpec}\,\mathbb{R}} \mathrm{AnSpec}\,\mathrm{Cont}(S,\mathbb{R})$. For each $s \in S$, evaluation at s gives a map $\mathrm{Cont}(S,\mathbb{R}) \to \mathbb{R}$, inducing $\mathrm{AnSpec}\,\mathbb{R} \to \mathrm{AnSpec}\,\mathrm{Cont}(S,\mathbb{R})$ and thus $X_\mathbb{R} \to X_\mathbb{R} \times_{\mathrm{AnSpec}\,\mathbb{R}} \mathrm{AnSpec}\,\mathrm{Cont}(S,\mathbb{R})$, pullback along which gives a line bundle $\overline{\mathcal{L}}_s$ on $X_\mathbb{R}$, where we know line bundles are classified by their degrees. This gives a map $S \to \mathbb{Z}$ sending

$$s \mapsto \deg \overline{\mathcal{L}}_s,$$

which must be locally constant, and therefore we can assume in fact constant, and by twisting we can assume is 0, i.e. $\overline{\mathcal{L}}_s \simeq \mathcal{O}_{X_\mathbb{R}}$ for each $s \in S$. Then

$$R\Gamma(X_\mathbb{R} \times_{\mathrm{AnSpec}\,\mathbb{R}} \mathrm{AnSpec}\,\mathrm{Cont}(S,\mathbb{R}), \overline{\mathcal{L}})$$

is a perfect complex of $\mathrm{Cont}(S,\mathbb{R})$-modules, a priori concentrated in degrees 0 and 1; but its first cohomology is fiberwise $H^1(X_\mathbb{R}, \mathcal{O}) = 0$, hence vanishes globally, so this is a line bundle concentrated in degree 0 over $\mathrm{Cont}(S,\mathbb{R})$, hence trivial, admitting a global section; therefore $\overline{\mathcal{L}}$ itself admits a global section and so is trivial.

We then want to trivialize $\mathcal{L}$: this amounts to extending the global section to the point at infinity $\mathrm{AnSpec}\,A$. If we assume that A is strongly totally disconnected, so $A^{\dagger\text{-red}} \simeq \mathrm{Cont}(S,\mathbb{C})$, then the section on $\mathrm{Cont}(S,\mathbb{C})$ automatically lifts to one on A; and we can assume this since (up to the filtered colimit issues) these essentially give a basis for totally disconnected algebras. ☐

3. Degree 1 divisors

We can now define the stack of degree 1 divisors rather straightforwardly: Div^1 is the object of TDStack sending a totally disconnected $\mathbb{C}$-algebra A to the anima of pairs $(\mathcal{L}, s)$ where $\mathcal{L}$ is a degree 1 line bundle on $X_{\mathbb{R},A}$ and $s \in \Gamma(X_{\mathbb{R},A}, \mathcal{L})$ is a fiberwise nonzero section, i.e. a section which is nonzero after pullback along every geometric point $\mathrm{AnSpec}\,\mathbb{C}_{\mathrm{gas}} \to A$. This is equivalent to giving a Cartier divisor $V(s) \to X_{\mathbb{R},A}$ (the vanishing locus of s); in the p-adic analogue these degree 1 divisors would be given by untilts of A. One of the main things missing from the archimedean picture is an independent interpretation of these "untilts."

By Theorem 10, the degree 1 line bundle $\mathcal{L}$ is locally isomorphic to $\mathcal{O}(1)$, so we want to study sections (up to scaling) of $\mathcal{O}(1)$ which are fiberwise nonzero. This suggests studying the subfunctor $\mathcal{BC}(\mathcal{O}(1)) \setminus \{0\} \subset \mathcal{BC}(\mathcal{O}(1))$, sending A to the anima of fiberwise nonzero sections of $\mathcal{O}_{X_{\mathbb{R},A}}(1)$. Since the last condition is determined by $\mathbb{C}$-points, the subfunctor is determined by the underlying condensed anima, and since the condition is open it there gives an open immersion $\mathcal{BC}(\mathcal{O}(1)) \setminus \{0\} \subset \mathcal{BC}(\mathcal{O}(1))$. Finally, accounting for scaling, we arrive at the following proposition:

Proposition 11. *There is an isomorphism*

$$\mathrm{Div}^1 \simeq (\mathcal{BC}(\mathcal{O}(1)) \setminus \{0\})/\mathbb{R}^{\times,\mathrm{la}}$$

in TDStack*.*

To make this into an explicit description of the stack Div^1, we would like to have an explicit description of $\mathcal{BC}(\mathcal{O}(1))$. First, we have to specify what we mean by $\mathcal{O}(1)$ rather than only its isomorphism class: take it to be the ample line bundle $\mathcal{O}([\infty])$ associated to the degree 1 divisor $\infty = \mathrm{AnSpec}\,\mathbb{C} \subset X_{\mathbb{R}}$.

Proposition 12. *There is an isomorphism*

$$\mathcal{BC}(\mathcal{O}(1)) \simeq \mathbb{A}^{1,\mathrm{an}} \times \mathbb{R}_{\mathrm{Betti}}$$

of $\mathcal{BC}(\mathcal{O}) \simeq \mathbb{R}^{\mathrm{la}}$*-module objects in* TDStack*.*

Proof. By the definition of $X_{\mathbb{R},A}$ by pushout, we have a Cartesian diagram

$$\begin{array}{ccc} \mathcal{BC}(\mathcal{O}(1))(A) & \longrightarrow & \Gamma(X_{\mathbb{R}}, \mathcal{O}(1)) \otimes_{\mathbb{R}} \mathrm{Cont}(S, \mathbb{R}) \\ \downarrow & & \downarrow \\ A & \longrightarrow & \mathrm{Cont}(S, \mathbb{C}) \end{array}$$

which as A (and thus S) varies gives

$$\mathcal{BC}(\mathcal{O}(1)) \simeq \mathbb{A}^{1,\mathrm{an}} \times_{\mathbb{C}_{\mathrm{Betti}}} H^0(X_{\mathbb{R}}, \mathcal{O}(1))_{\mathrm{Betti}}.$$

By Lemma 4.1.5 of Jaburi's master's thesis, $H^0(X_{\mathbb{R}}, \mathcal{O}(1)) \simeq \mathbb{R}^3$, and pullback along the point at infinity gives a surjection $H^0(X_{\mathbb{R}}, \mathcal{O}(1)) \to \mathbb{C} \simeq \mathbb{R}^2$, so that choosing a splitting gives

$$\mathcal{BC}(\mathcal{O}(1)) \simeq \mathbb{A}^{1,\mathrm{an}} \times_{\mathbb{C}_{\mathrm{Betti}}} (\mathbb{C}_{\mathrm{Betti}} \times \mathbb{R}_{\mathrm{Betti}}) \simeq \mathbb{A}^{1,\mathrm{an}} \times \mathbb{R}_{\mathrm{Betti}}$$

as claimed. □

In fact, we can be a little more precise: the map $\mathbb{R}^{\mathrm{la}} \simeq \mathcal{BC}(\mathcal{O}) \hookrightarrow \mathcal{BC}(\mathcal{O}[\infty])$ factors through the kernel of $H^0(X_{\mathbb{R}}, \mathcal{O}(1))_{\mathrm{Betti}} \to \mathbb{C}_{\mathrm{Betti}}$, which by the discussion above is $\mathbb{R}_{\mathrm{Betti}}$. On the other hand taking the fiber at infinity gives a surjection $\mathcal{BC}(\mathcal{O}(1)) \to \mathbb{A}^{1,\mathrm{an}}$ given on A-points by the left vertical map above $\mathcal{BC}(\mathcal{O}(1))(A) \to A$, whose kernel is then $\mathbb{R}_{\mathrm{Betti}}$. This gives a short exact sequence

$$0 \to \mathbb{R}_{\mathrm{Betti}} \to \mathcal{BC}(\mathcal{O}(1)) \to \mathbb{A}^{1,\mathrm{an}} \to 0$$

of $\mathbb{R}^{\mathrm{la}}$-module objects in TDStack, which splits on $\mathbb{C}$-valued point, or equivalently on Betti stacks, and thus splits in TDStack. An explicit splitting is given by identifying $\mathbb{C}$-points of $\mathcal{BC}(\mathcal{O}(1))$, after pullback along $\mathbb{G}_m \to X_{\mathbb{R}}$ (so away from 0 and ∞), with functions $az + b + \overline{a}z^{-1}$ for $a \in \mathbb{C}$ and $b \in \mathbb{R}$ corresponding to $(a, b) \in (\mathbb{A}^{1,\mathrm{an}} \times \mathbb{R}_{\mathrm{Betti}})(\mathbb{C})$. One can check that this splitting is U(1)-equivariant, and in fact is the unique U(1)-equivariant splitting.

So in summary we have obtained an explicit description of Div^1 as $((\mathbb{A}^{1,\mathrm{an}} \times \mathbb{R}_{\mathrm{Betti}}) \setminus \{0\})/\mathbb{R}^{\times,\mathrm{la}}$. Our next goal will be to relate vector bundles on Div^1 to representations of the real Weil group $W_{\mathbb{R}}$; however from this presentation it is not clear how this should work. Therefore we'll first see if we can give a second presentation for Div^1 which makes the connection with the Weil group clearer.

4. Working over $\mathbb{C}$

Throughout this course, we have generally worked over $\mathbb{R}$ as our archimedean local field of interest, though sometimes with complex coefficients. However there is of course another archimedean local field, namely the complex numbers. Most of the story over $\mathbb{C}$ is very similar to over $\mathbb{R}$, and indeed often simpler, e.g. the (absolute!) archimedean Fargues–Fontaine curve over $\mathbb{C}$ is just $\mathbb{P}^1_{\mathbb{C}}$, and one can generally recover everything by pulling back from the version over $\mathbb{R}$. However it will be useful for us to make some of the story explicit over $\mathbb{C}$: in particular we will eventually obtain a second description of Div^1 which will be useful to us for understanding representations of the Weil group.

First, we need to define the relative complex Fargues–Fontaine curve in families. We do this by pullback:

$$X_{\mathbb{C},A} = X_{\mathbb{R},A} \times_{\mathrm{AnSpec}\,\mathbb{R}} \mathrm{AnSpec}\,\mathbb{C}.$$

Explicitly, bearing in mind that the preimage of the complex point at infinity is now two points $\{0\} \sqcup \{\infty\}$ in $\mathbb{P}^1_{\mathbb{C}}$, the relative curve is given by the pushout

$$\begin{array}{ccc} \mathrm{AnSpec}\,\mathrm{Cont}(S,\mathbb{C}) \sqcup \mathrm{AnSpec}\,\mathrm{Cont}(S,\mathbb{C}) & \xrightarrow{0\sqcup\infty} & \mathbb{P}^1_{\mathbb{C}} \times_{\mathrm{AnSpec}\,\mathbb{C}} \mathrm{Cont}(S,\mathbb{C}) \\ \downarrow & & \downarrow \\ \mathrm{AnSpec}(A \otimes_{\mathbb{C}, z\mapsto\overline{z}} \mathbb{C}) \sqcup \mathrm{AnSpec}(A) & \longrightarrow & X_{\mathbb{C},A} \end{array}.$$

We can then define $\mathrm{Pic}_{\mathbb{C}}$ as the stack sending A to line bundles on $X_{\mathbb{C},A}$; exactly as for real numbers we verify that this is a totally disconnected stack and that there is an isomorphism

$$\bigsqcup_{n\in\mathbb{Z}} */\mathbb{C}^{\times,\mathrm{la}} \to \mathrm{Pic}_{\mathbb{C}},$$

giving rise to a degree map $\deg : \mathrm{Pic}_{\mathbb{C}} \to \mathbb{Z}$ sending $\mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(n) \mapsto n$. Note that this is not (naively) compatible with pullback of line bundles from the real case: if $\nu_A : X_{\mathbb{C},A} \to X_{\mathbb{R},A}$ is the covering map and $\nu^* : \mathrm{Pic} \to \mathrm{Pic}_{\mathbb{C}}$ the induced pullback, then $\deg \circ \nu^* = 2\deg$.

We can similarly define $\mathrm{Div}^1_{\mathbb{C}}$, where we do have a well-defined *pushforward* of divisors $\mathrm{Div}^1_{\mathbb{C}} \to \mathrm{Div}^1$, sending $V(\widetilde{s}) \to X_{\mathbb{C},A}$ to its composite $V(\widetilde{s}) \to X_{\mathbb{C},A} \to X_{\mathbb{R},A}$. This is invariant under complex conjugation, suggesting the following proposition:

Proposition 13. *The natural map*

$$\mathrm{Div}^1_{\mathbb{C}} / \mathrm{Gal}(\mathbb{C}/\mathbb{R}) \to \mathrm{Div}^1$$

is an isomorphism.

Proof. The fibers of $\mathrm{Div}^1_{\mathbb{C}} \to \mathrm{Div}^1$ are lifts of divisors $V(s)$ to $\mathbb{C}$. Since $X_{\mathbb{R},A}$ locally lifts to $\mathbb{C}$, so we can always find lifts locally, and any two local lifts are Galois conjugate. □

Completely parallel to Proposition 11, we have the following description of $\mathrm{Div}^1_{\mathbb{C}}$:

Proposition 14. *There is an isomorphism*

$$\mathrm{Div}^1_{\mathbb{C}} \simeq (\mathcal{BC}_{\mathbb{C}}(\mathcal{O}(1)) \setminus \{0\})/\mathbb{C}^{\times,\mathrm{la}}$$

in TDStack, *where $\mathcal{BC}_{\mathbb{C}}$ is the variant of $\mathcal{BC}$ replacing $X_{\mathbb{R},A}$ by $X_{\mathbb{C},A}$.*

We then want to describe $\mathcal{BC}_{\mathbb{C}}(\mathcal{O}(1))$:

Proposition 15. *There is an isomorphism*

$$\mathcal{BC}_{\mathbb{C}}(\mathcal{O}(1)) \simeq \mathbb{A}^{1,\mathrm{an}} \times \mathbb{A}^{1,\mathrm{an}}$$

of $\mathcal{BC}_{\mathbb{C}}(\mathcal{O}) \simeq \mathbb{C}^{\mathrm{la}}$-module objects, where $z \in \mathbb{C}^{\mathrm{la}}$ acts on $\mathbb{A}^{1,\mathrm{an}} \times \mathbb{A}^{1,\mathrm{an}}$ by $z \cdot (t_1, t_2) = (zt_1, \overline{z}t_2)$.

Proof. Similarly to the proof of Proposition 12, we obtain from the pushout description of $X_{\mathbb{C},A}$ the formula

$$\mathcal{BC}_{\mathbb{C}}(\mathcal{O}(1)) \simeq (\mathbb{A}^{1,\mathrm{an}} \times \mathbb{A}^{1,\mathrm{an}}) \times_{(\mathbb{C}\times\mathbb{C})_{\mathrm{Betti}}} H^0(\mathbb{P}^1_{\mathbb{C}}, \mathcal{O}(1))_{\mathrm{Betti}}.$$

In this case $H^0(\mathbb{P}^1_{\mathbb{C}}, \mathcal{O}(1)) \simeq \mathbb{C} \times \mathbb{C}$ and in particular the map $H^0(\mathbb{P}^1_{\mathbb{C}}, \mathcal{O}(1)) \to \mathbb{C} \times \mathbb{C}$ given by taking the fibers at 0 and ∞ is an isomorphism, so we recover the stated formula. The action of $\mathbb{C}^{\times,\mathrm{la}}$ comes from the twist by complex conjugation on one factor of $\mathbb{A}^{1,\mathrm{la}}(A) = A$ in the pushout definition of $X_{\mathbb{R},A}$. □

Again, we can be a little more precise: taking the fibers at 0 and ∞ gives an isomorphism

$$\mathcal{BC}_{\mathbb{C}}(\mathcal{O}(1)) \simeq \mathbb{A}^{1,\mathrm{an}} \times \mathbb{A}^{1,\mathrm{an}}$$

which on $\mathbb{C}$-points (away from 0) identifies points of $\mathcal{BC}_{\mathbb{C}}(\mathcal{O}(1))$ with sections $az+\overline{b}$ of $\mathcal{O}([\infty])$ for $(a, b) \in (\mathbb{A}^{1,\mathrm{an}} \times \mathbb{A}^{1,\mathrm{an}})(\mathbb{C})$.

We can identify $\mathbb{A}^{1,\mathrm{an}} \times \mathbb{A}^{1,\mathrm{an}}$ with $\mathbb{A}^{2,\mathrm{an}}$ with a nonstandard action of $\mathbb{C}^{\mathrm{la}}$ (via the twist by complex conjugation). This in turn can also be identified with $\mathcal{BC}(\mathcal{O}(1/2))$, where $\mathcal{O}(1/2) = \nu_*\mathcal{O}(1)$, by similar arguments, which has automorphism group $\mathbb{H}^\times$; in particular embedding $\mathbb{C}^\times \subset \mathbb{H}^\times$ we get the right action of $\mathbb{C}^\times$ on $\mathbb{A}^{2,\mathrm{an}}$.

Combining Propositions 13, 14, and 15, we obtain another description of Div^1 as the quotient of the punctured plane by two groups: first by $\mathbb{C}^{\times,\mathrm{la}}$, and second by $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$. Recalling that the real Weil group $W_{\mathbb{R}} \subset \mathbb{H}^\times$ is the combination of these two groups, we might guess the following description:

Corollary 16. *There is an isomorphism*

$$\mathrm{Div}^1 \simeq (\mathbb{A}^{2,\mathrm{an}} \setminus \{0\})/W_{\mathbb{R}}^{\mathrm{la}}$$

where $W_{\mathbb{R}} \subset \mathbb{H}^\times$ acts on $\mathbb{A}^2 \simeq \mathbb{H} \otimes_{\mathbb{C}} \mathbb{A}^{1,\mathrm{an}}$ by multiplication on the left factor.

Proof. Fix an embedding $\mathbb{C} \subset \mathbb{H} = \mathrm{End}(\mathcal{O}(1/2))$ as a maximal commutative subalgebra. For a fiberwise nonzero section $s : \mathcal{O} \to \mathcal{O}(1/2)$, base changing to $\mathbb{C}$ gives a map $\mathbb{C} \otimes_{\mathbb{R}} \mathcal{O} \to \mathcal{O}(1/2)$ which is a degree 1 embedding of vector bundles and so defines a degree 1 divisor as its cokernel. This gives a map

$$\mathcal{BC}(\mathcal{O}(1/2)) \setminus \{0\} \to \mathrm{Div}^1 .$$

It is invariant under the action of the normalizer of $\mathbb{C}$ in $\mathbb{H}$, which is precisely $W_{\mathbb{R}}$; this can be understood as the combination of the invariance under $\mathbb{C}^\times$-scaling and Galois conjugation of Propositions 13 and 14. Thus we get a map

$$(\mathcal{BC}(\mathcal{O}(1/2)) \setminus \{0\})/W_{\mathbb{R}}^{\mathrm{la}} \to \mathrm{Div}^1 .$$

Identifying $\mathcal{BC}(\mathcal{O}(1/2))$ with $\mathbb{A}^{2,\mathrm{an}}$ as above, we see that in fact this map must be an isomorphism and gives the claimed description of Div^1. □

This is now much more amenable to the kind of relationship we want to study: vector bundles on Div^1 are now manifestly equivalent to $W_{\mathbb{R}}$-equivariant vector bundles on $\mathbb{A}^{2,\mathrm{an}} \setminus \{0\}$. In the next section, we'll see how these more directly relate to locally analytic $W_{\mathbb{R}}$-representations.

First, though, we want to make explicit the relationship between the two descriptions of Div^1 as $((\mathbb{A}^{1,\mathrm{an}} \times \mathbb{R}_{\mathrm{Betti}}) \setminus \{0\})/\mathbb{R}^{\times,\mathrm{la}}$ and $(\mathbb{A}^{2,\mathrm{an}} \setminus \{0\})/W_{\mathbb{R}}^{\mathrm{la}}$. In the language of archimedean Banach–Colmez spaces, this can be interpreted as follows: for a section s of $\mathcal{BC}(\mathcal{O}(1/2))$, or equivalently of $\mathcal{BC}_{\mathbb{C}}(\mathcal{O}(1))$, we want to associate to it a section t of $\mathcal{BC}(\mathcal{O}(1))$ with the same vanishing locus. This will then give the isomorphism from the second description to the first.

Explicitly, a section s of $\mathcal{BC}_{\mathbb{C}}(\mathcal{O}(1))$ is a section $az+\bar{b}$ for $(a,b) \in (\mathbb{A}^{1,\mathrm{an}} \times \mathbb{A}^{1,\mathrm{an}}) \setminus \{(0,0)\})$, which we can map to

$$(az + \bar{b})(-\bar{a}z^{-1} + b) = abz + (b\bar{b} - a\bar{a}) - \bar{a}\bar{b}z^{-1}$$

which corresponds to $(ab, b\bar{b} - a\bar{a}) \in (\mathbb{A}^{1,\mathrm{an}} \times \mathbb{R}_{\mathrm{Betti}}) \setminus \{0\}$. There are two copies of $\mathbb{C}^{\times,\mathrm{la}}$ inside $W_{\mathbb{R}}^{\mathrm{la}}$, and one can check that the prescribed action of $z \in \mathbb{C}^\times$ on $\mathbb{A}^{2,\mathrm{an}}$ induces actions by $z\bar{z}$ and $-z\bar{z}$ respectively for these two copies on $\mathbb{A}^{1,\mathrm{an}} \times \mathbb{R}_{\mathrm{Betti}}$. One can also give a similar formula in the other direction.

5. L-parameters and Div^1

Our final goal for today is to start the discussion of how L-parameters are related to the stack Div^1. We have seen that we can identify Div^1 with $(\mathbb{A}^{2,\mathrm{an}} \setminus \{0\})/W_{\mathbb{R}}^{\mathrm{la}}$, and so there is a natural projection $\mathrm{Div}^1 \to */W_{\mathbb{R}}^{\mathrm{la}}$. Pullback therefore induces a functor $D(*/W_{\mathbb{R}}^{\mathrm{la}}) \to D(\mathrm{Div}^1)$. To say more, it would be convenient if we could work with $\mathbb{A}^{2,\mathrm{an}}/W_{\mathbb{R}}^{\mathrm{la}}$ rather than the quotient of the punctured plane, since then the origin gives a canonical section $* \to \mathbb{A}^{2,\mathrm{an}}$ giving rise to a factorization of the identity

$$*/W_{\mathbb{R}}^{\mathrm{la}} \xrightarrow{s} \mathbb{A}^{2,\mathrm{an}}/W_{\mathbb{R}}^{\mathrm{la}} \xrightarrow{\pi} */W_{\mathbb{R}}^{\mathrm{la}} .$$

One of our main results next time will be the following theorem saying that we can use these perspectives interchangeably:

Theorem 17. *Every vector bundle on* $\mathrm{Div}^1 \simeq (\mathbb{A}^{2,\mathrm{an}}\backslash\{0\})/W_{\mathbb{R}}^{\mathrm{la}}$ *extends uniquely to* $\mathbb{A}^{2,\mathrm{an}}/W_{\mathbb{R}}^{\mathrm{la}}$.

In particular on the level of isomorphism classes, vector bundles on $*/W_{\mathbb{R}}^{\mathrm{la}}$*, i.e. locally analytic* $W_{\mathbb{R}}$*-representations, embed into vector bundles on* Div^1 *via* π^**. If* s^*V *is semisimple as a* $W_{\mathbb{R}}$*-representation for a vector bundle* V *on* Div^1*, then* V *is in the image of* π^**, inducing a bijection between isomorphism classes of semisimple* $W_{\mathbb{R}}$*-representations and semisimple vector bundles on* Div^1.

More generally, we want to study L-parameters, which for (say) a split group G are classically given by maps $\varphi : W_{\mathbb{R}} \to \widehat{G}(\mathbb{C})$. This suggests that L-parameters for G should correspond to $\widehat{G}$-torsors on Div^1; this is what we will generally mean by L-parameters in our geometrization going forward.

In general, though, there are $\widehat{G}$-torsors on Div^1 which do not correspond to any classical L-parameters. This is actually a feature, not a bug: archimedean L-parameters are often poorly behaved in families, which is a major issue with the archimedean local Langlands program. For example, while as $G(\mathbb{R})$-representations the principal and discrete series representations interact,[1] their L-parameters are in some sense in different components of the parameter space. One possible solution is to allow "refined" parameter spaces in which degenerations from discrete to principal series L-parameters exist, as introduced by Adams–Barbasch–Vogan. We will see (next time) that these refined parameters can also be related to $\widehat{G}$-torsors on Div^1.

For now, we focus on a simpler piece of the story. We are interested in studying vector bundles on $\mathrm{Div}^1 \simeq (\mathbb{A}^{2,\mathrm{an}} \setminus \{0\})/W_{\mathbb{R}}^{\mathrm{la}}$. To make things easier, we can look at the cover $\mathrm{Div}^1_{\mathbb{C}} \simeq (\mathbb{A}^{2,\mathrm{an}} \setminus \{0\})/\mathbb{C}^{\times,\mathrm{an}}$, or even simpler inside $\mathbb{A}^{2,\mathrm{an}} \setminus \{0\}$ we have the open locus $\mathbb{G}_{\mathrm{m}}^{2,\mathrm{an}}$ where the $\mathbb{C}^{\times,\mathrm{la}}$-action is free, giving an identification

$$\mathbb{G}_{\mathrm{m}}^{2,\mathrm{an}}/\mathbb{C}^{\times,\mathrm{la}} \simeq \mathbb{C}^{\times}_{\mathrm{Betti}}.$$

On the other hand the natural projection $\mathbb{G}_{\mathrm{m}}^{2,\mathrm{an}}/\mathbb{C}^{\times,\mathrm{la}} \to */\mathbb{C}^{\times,\mathrm{la}}$ gives a map

$$\mathbb{C}^{\times}_{\mathrm{Betti}} \to */\mathbb{C}^{\times,\mathrm{la}}.$$

Vector bundles on the target are locally analytic $\mathbb{C}^{\times}$-representations; vector bundles on the source are local systems on $\mathbb{C}^{\times}$, which are determined by their monodromy.

We briefly review the representation theory of $\mathbb{C}^{\times}$ in a convenient language. First of all, the representation theory of $\mathbb{C}$ as an additive group is simple: all irreducible representations are one-dimensional, and of the form $z \mapsto \exp(\lambda_1 z + \lambda_2 \overline{z})$ for $\lambda_1, \lambda_2 \in \mathbb{C}$. To relate this to $\mathbb{C}^{\times}$, we use the exponential sequence

$$0 \to 2\pi i\mathbb{Z} \to \mathbb{C} \xrightarrow{\exp} \mathbb{C}^{\times} \to 0$$

so a character of $\mathbb{C}$ descends to one of $\mathbb{C}^{\times}$ if and only if $\lambda_1 - \lambda_2 \in \mathbb{Z}$, i.e. characters of $\mathbb{C}^{\times}$ correspond to pairs (λ_1, λ_2) of complex numbers where $\lambda_1 - \lambda_2 \in \mathbb{Z}$. In particular we associate to any character χ (corresponding to some such (λ_1, λ_2)) the invariant $\alpha = \exp(2\pi i\lambda_1) = \exp(2\pi i\lambda_2)$.

[1] I would like to understand this better—I believe Scholze talks about it in his first Noether lecture on this topic, but I haven't had time to go look back; I'll edit this if I do.

We claim that for a character χ of $\mathbb{C}^\times$, viewed as a line bundle on $*/\mathbb{C}^{\times,\mathrm{la}}$, the pullback to $\mathbb{C}^\times_{\mathrm{Betti}}$ is the local system with monodromy $\alpha = \alpha(\chi)$.

Indeed, take an embedding $\mathbb{C}^{\times,\mathrm{la}} \subset \mathbb{G}_{\mathrm{m}}^{2,\mathrm{an}}$ and consider a $\mathbb{Z}$-cover $G \to \mathbb{G}_{\mathrm{m}}^{2,\mathrm{an}}$ whose fundamental group agrees with that of $\mathbb{C}^{\times,\mathrm{la}}$ inside $\mathbb{G}_{\mathrm{m}}^{2,\mathrm{an}}$, so that the embedding $\mathbb{C}^{\times,\mathrm{la}} \subset \mathbb{G}_{\mathrm{m}}^{2,\mathrm{an}}$ lifts to $\mathbb{C}^{\times,\mathrm{la}} \subset G$. The quotient $G/\mathbb{C}^{\times,\mathrm{la}}$ is $\mathbb{C}_{\mathrm{Betti}}$, so we have a Cartesian diagram

$$\begin{array}{ccc} \mathbb{C}_{\mathrm{Betti}} & \longrightarrow & * \\ \downarrow & & \downarrow \\ */\mathbb{C}^{\times,\mathrm{la}} & \longrightarrow & */G \end{array}$$

so since $\mathbb{C}$ is contractible pullback along the lower map induces an equivalence of vector bundles. In particular, any character χ of $\mathbb{C}^{\times,\mathrm{la}}$ extends uniquely to a character of G.

The fiber of G over $(z_1, z_2) \in \mathbb{G}_{\mathrm{m}}^{2,\mathrm{an}}$ is isomorphic to $\mathbb{Z}$; for a character χ associated to the pair (λ_1, λ_2), its evaluation on elements of the kernel $(1, 1, n)$ is $\exp(2\pi i\lambda_1 n) = \exp(2\pi i\lambda_2 n)$, and so in particular on a generator of the kernel we recover α. On the other hand the $\mathbb{Z}$-cover

$$G/\mathbb{C}^{\times,\mathrm{la}} \simeq \mathbb{C}_{\mathrm{Betti}} \to \mathbb{G}_{\mathrm{m}}^{2,\mathrm{an}}/\mathbb{C}^{\times,\mathrm{la}} \simeq \mathbb{C}^\times_{\mathrm{Betti}}$$

is (the Betti stack version of) the exponential map for which the action of the generator of the kernel is exactly the monodromy.

More generally, a representation $\varphi : \mathbb{C}^\times \to \mathrm{GL}_n(\mathbb{C})$ is determined by two commuting $n \times n$-matrices λ_1, λ_2 over $\mathbb{C}$ such that $\exp(2\pi i\lambda_1) = \exp(2\pi i\lambda_2) =: \alpha \in \mathrm{GL}_n(\mathbb{C})$, which can similarly be thought of as the monodromy of the pullback of the corresponding vector bundle on $*/\mathbb{C}^{\times,\mathrm{la}}$ to $\mathbb{C}^\times_{\mathrm{Betti}}$. We can show that φ is semisimple if and only if α is: if φ is semisimple, it is a sum of characters and so α is as well by inspection. On the other hand if φ is not semisimple, it is the extension of some set of characters; but two distinct characters of $\mathbb{C}^\times$ have no extensions, so it must be the extension of a character by itself (some number of times), which we can assume by twisting to be the trivial character. Thus in this situation φ is an extension of direct sums of the trivial character, so λ_1 and λ_2 must be nilpotent; but then the exponential map is injective on them, so $\lambda_1 = \lambda_2$ are determined by α, and would have to be trivial if α were semisimple.

Thus we can relate vector bundles on the large open subspace $\mathbb{G}_{\mathrm{m}}^{2,\mathrm{an}}/\mathbb{C}^{\times,\mathrm{la}} \subset \mathrm{Div}^1_{\mathbb{C}}$ to $\mathbb{C}^{\times,\mathrm{la}}$-representations. Next time, we will study how we can extend this to the whole space, and the descent to Div^1. This already involves studying these geometric spaces via analytic tools such as T-connections, which we can understand as a transmutation interpretation; time permitting we will begin to look further into how spaces such as Div^1 and its relatives can be used to classify such structures and others, including relationships to twistor and Hodge structures.